The Radiance Equation

Jan Kautz

Outline

- Basic terms in radiometry
- Radiance
- Reflectance
- The Radiance Equation
- The operator form of the radiance equation
- Meaning of the operator form
- Approximations to the radiance equation

Light: Radiant Power

- Φ denotes the *radiant energy* or *flux* in a volume *V*.
- The flux is the rate of energy flowing through a surface per unit time (watts).
- The energy is proportional to the particle flow, since each photon carries energy.
- The flux may be thought of as the flow of photons per unit time.

Light: Flux Equilibrium

- Total flux in a volume in dynamic equilibrium
  - Particles are flowing
  - Distribution is constant
- Conservation of energy
  - Total energy input into the volume = total energy that is output by or absorbed by matter within the volume.

Light: Equation

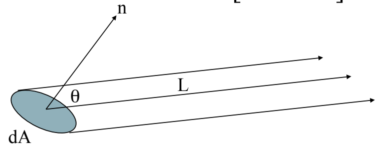
- $\Phi(p,\omega)$ denotes flux at $p \in V$, in direction ω
- It is possible to write down an integral equation for $\Phi(p,\omega)$ based on:
  - Emission+Inscattering = Streaming+Outscattering + Absorption
- Complete knowledge of $\Phi(p,\omega)$ provides a complete solution to the graphics rendering problem.
- Rendering is about solving for $\Phi(p,\omega)$.

Radiance

- Radiance (*L*) is the flux that leaves a surface, per unit projected area of the surface, per unit solid angle of direction.

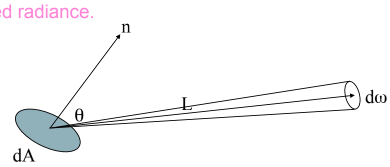
$$d\Phi = \left[\int L \cos\theta \, d\omega\right] dA$$

$$L = d^2\Phi \,/\, \left[\cos\theta \, d\omega \, dA\right]$$



Radiance

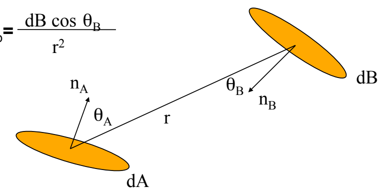
- For computer graphics the basic particle is not the photon and the energy it carries but the ray and its associated radiance.



Radiance is constant along a ray.

Solid angle

$$d\omega_b = \frac{dB \cos\theta_B}{r^2}$$



Radiance: Radiosity, Irradiance

- Radiosity - is the flux per unit area that radiates from a surface, denoted by *B*.
  - dΦ = B dA
- Irradiance is the flux per unit area that arrives at a surface, denoted by *E*.
  - dΦ = E dA

Radiosity and Irradiance

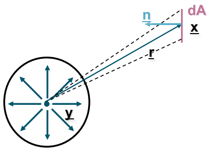
- L(p,ω) is radiance at p in direction ω
- E(p) is irradiance at p
- E(p) = (dΦ/dA) = ∫ L(p,ω) cosθ dω
- (or: L = dE/dA)

Light Sources – Point Light

- Point light with isotropic radiance
  - Power (total flux) of a point light source
    - Φ_s = Power of the light source [Watt]
  - Intensity of a light source
    - $I = \Phi_s/(4\pi\, sr)$ [Watt/sr]
  - Irradiance on a sphere with radius *r* around light source:
    - $E_r = \Phi_s/(4\pi r^2)$ [Watt/m²]
  - Irradiance on a surface A

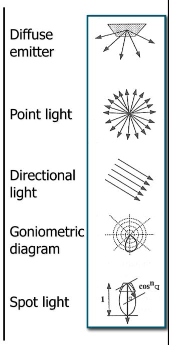
$$E(x) = \frac{d\Phi_s}{dA} = I\frac{d\omega}{dA} = \frac{\Phi_s}{4\pi}\cdot\frac{dA\cos\theta}{r^2 dA} = \frac{\Phi_s}{4\pi}\cdot\frac{\cos\theta}{r^2}$$

$$\underline{r} = \underline{x} - \underline{y}$$



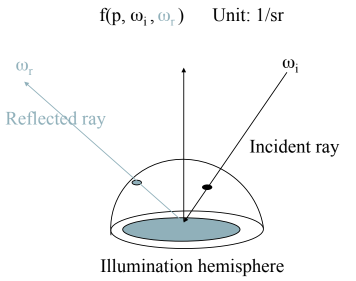
Light Sources

- Other types of light sources
  - Spot-lights
    - Cone of light
    - Radiation characteristic of $\cos^n\theta$
  - Area light sources
  - Point light sources with non-uniform directional power distribution
- Other parameter
  - Atmospheric attenuation with distance (r) for point light sources
    - $1/(ar^2+br+c)$
    - Physically correct would be $1/r^2$
    - Correction of missing ambient light



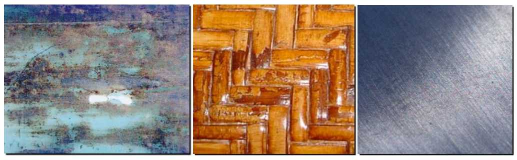
Reflectance

- BRDF
  - Bi-directional
  - Reflectance
  - Distribution
  - Function
- Relates
  - Reflected radiance to incoming irradiance



BRDF

- Boils down to: How much light is reflected for a given light/ view direction at a point?
- Defines the "look" of the surface
- Important part for realistic surfaces:
  - Variation (in texture, gloss, …)



Properties of BRDFs

- Non-negativity

$$f_r(\theta_i,\phi_i,\theta_r,\phi_r) \geq 0$$

- Energy Conservation

$$\int_\Omega f_r(\theta_i,\phi_i,\theta_r,\phi_r) d\mu(\theta_r,\phi_r) \leq 1 \quad \text{for all } (\theta_i,\phi_i)$$

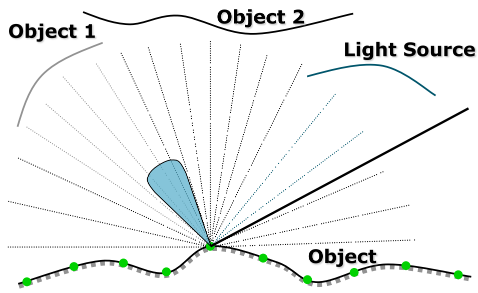
- Reciprocity

$$f_r(\theta_i,\phi_i,\theta_r,\phi_r) = f_r(\theta_r,\phi_r,\theta_i,\phi_i)$$

  - Aside: actually does often **not** hold for real materials [see Eric Veach's PhD Thesis!]

How to compute reflected light?

- Integrate all incident light * BRDF



Reflectance: BRDF

- Reflected Radiance =

$L(p, \omega_r) = \int f(p, \omega_i, \omega_r)\, L(p, \omega_i) \cos\theta_i \, d\omega_i$

- In practice BRDF's are hard to specify
- Commonly rely on ideal types
  - Perfectly diffuse reflection
  - Perfectly specular reflection
  - Glossy reflection
- BRDFs taken as additive mixture of these

The Radiance Equation

- Radiance $L(p, \omega)$ at a point p in direction ω is the sum of
  - Emitted radiance $L_e(p, \omega)$
  - Total reflected radiance

Radiance = Emitted Radiance + Total Reflected Radiance

The Radiance Equation: Reflection

- Total reflected radiance in direction ω:

$$\int f(p, \omega_i, \omega)\, L(p^*, -\omega_i) \cos\theta_i \, d\omega_i$$

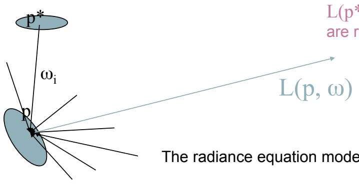
(p* is closest point in direction ω_i)

- Full Radiance Equation:
- $L(p, \omega) = L_e(p, \omega) + \int f(p, \omega_i, \omega)\, L(p^*, -\omega_i) \cos\theta_i \, d\omega_i$
  - (Integration over the illumination hemisphere)

The Radiance Equation

- p is considered to be on a surface, but can be anywhere, since radiance is constant along a ray, trace back until surface is reached at p', then
  - $L(p, \omega_i) = L(p', \omega_i)$

$L(p, \omega)$ depends on all $L(p^*, -\omega_i)$ which in turn are recursively defined.



The radiance equation models global illumination.

Operator form of the Radiance Equation

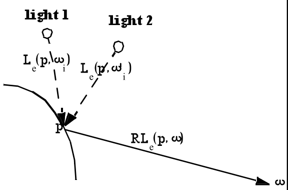
- Define the operator R to mean
- $(RL)(p, \omega) = \int f(p, \omega_i, \omega)\, L(p^*, -\omega_i) \cos\theta_i \, d\omega_i$
  - Use the notation $RL(p, \omega) = L^1(p, \omega)$
  - Repeated applications of R can be applied
  - $R(RL(p, \omega)) = R^2L(p, \omega) = RL^1(p, \omega) = L^2(p, \omega)$
  - …
- The operator 1 means the identity:
  - $1L(p, \omega) = L(p, \omega)$

Operator Form

- Using this notation, the radiance equation can be rewritten as:
  - $L = L_e + RL$
- We can rearrange this as:
  - $(1-R)L = L_e$
- Operator theory allows the normal algebraic operations:
  - $L = (1-R)^{-1}L_e$
  - $L = (1 + R + R^2 + R^3 + \ldots)\, L_e$ (Neumann series/expansion)

Meaning of the Operator

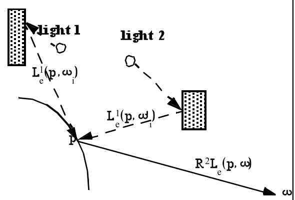
- $L_e(p, \omega_i)$ is radiance corresponding to direct lighting from a source (if any) from direction ω_i at point p.
- $RL_e(p, \omega_i)$ is therefore the radiance from point p in direction ω due to this direct lighting.



This is light that is 'one step removed' from the sources.

Meaning of the Operator

- $R^2L_e(p, \omega_i) = RL^1_e(p, \omega_i)$ is therefore light that is 'twice removed' from the light sources.
- Similar meanings can be attributed to $R^3L_e(p, \omega_i)$, $R^4L_e(p, \omega_i)$ and so on.



In general $R^iL_e(p, \omega_i)$ is the contribution to radiance from p in direction ω from all light paths of length i+1 back to the sources.

The Radiance Equation

- In general the radiance equation in operator form shows that L(p,ω) may be decomposed into light due to
  - The emissive properties of the surface at p
  - Plus that due directly to sources
  - Plus that reflected once from sources
  - Plus that reflected twice
  - … to infinity

Truncating the Equation

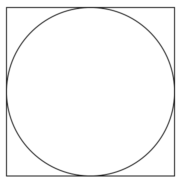
- Suppose the series is truncated after the first term $(1)L_e$
  - Only objects that are emitters would be shown
- Suppose one more term is added $(1+R)L_e$
  - Only direct lighting (and shadows) are accounted for.
- Suppose another term is added $(1+R+R^2)L_e$
  - Additionally one level of reflection is accounted for.
- …and so on.
- Each type of rendering method is a special case of this rendering equation, and computer graphics rendering consists of different types of approximation.

Monte Carlo Methods

- The radiance equation is an integral equation.
- Monte Carlo methods may be used to solve this.
- Monte Carlo methods involve using a random sampling technique to solve deterministic problems.

Simple Example - Finding π

- Choose a random sample of n uniformly distributed points in the square of side 2.
- Count how many (r) in the circle.
- $r/n \rightarrow \pi/4$ with prob. 1 as $n \rightarrow \infty$

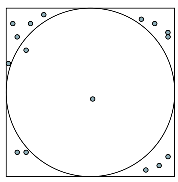


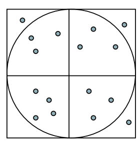
Variance Reduction

- The convergence rate of this procedure will be quite slow.
- The standard error of the estimator $\propto 1/\sqrt{n}$
- The problem in MC methods is to find ways to reduce the variance.
- A standard technique is stratified sampling.

Stratified Sampling

- A uniform random sample is *random*!
- Each type of pattern is equally probable.
- A stratified sample, where we sample randomly within strata significantly reduces the variance.





stratified

Example

$f(x) = e^{\sin(3x^2)}$

N	I
1	2.75039
10	1.9893
100	1.79139
1000	1.75146
10000	1.77313
100000	1.77862

Unstratified

$O(1/\sqrt{N})$

$f(x) = e^{\sin(3x^2)}$

N	I
1	2.70457
10	1.72858
100	1.77925
1000	1.77606
10000	1.77610
100000	1.77610

Stratified

$O(1/N)$

Slide borrowed from Henrik Wann Jensen

Antialiasing in Ray Tracing

- In order to reduce aliasing due to undersampling in ray tracing each pixel may be sampled and then the average radiance per pixel found.
- A stratified sample over the pixel is preferable to a uniform sample – especially when the gradient within the pixel is sharply changing.

Conclusion

- Radiance equation formally revisited
- And defined as an operator form
- Introduction to Monte Carlo sampling
- Applied to ray tracing